The Atiyah-Bott Localization Formula

in Enumerative Geometry

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Outline

References

- *Equivariant Cohomology in Algebraic Geometry* (David Anderson and William Fulton)
- *3264 and All That* (David Eisenbud and Joe Harris)

Definition

Suppose we have a group G acting on a topological space X. We can find a contractible space $\mathbb{E}G$ with a free right G-action. We then form the quotient space (call it the Borel product)

$$\mathbb{E}G \times^G X := \mathbb{E}G \times X / (e \cdot g, x) \sim (e, g \cdot x).$$

Definition

The equivariant cohomology of X with respect to G is

$$H_G^*(X) := H^*(\mathbb{E}G \times^G X; \mathbb{Z}).$$

Special case: If X is a point, then $H_G^*(\text{pt}) = H^*(\mathbb{B}G)$, where $\mathbb{B}G = \mathbb{E}G/G$ is the classifying space of G. Write $\Lambda_G = H_G^*(\text{pt})$

For a Torus...

Example

Consider a torus $T = (\mathbb{C}^*)^n$. Then $\mathbb{E}T = (\mathbb{C}^\infty - 0)^n$ and $\mathbb{B}T = (\mathbb{P}^\infty)^n$, and

$$\Lambda_T = \mathbb{Z}[t_1, \ldots, t_n],$$

where $t_i = c_1(\mathscr{O}_{\mathbb{P}^\infty}(-1))$ is the tautological line bundle on the i-th factor of $(\mathbb{P}^\infty)^n$.

Equivariant Chern Class

A vector bundle $E \to X$ is G-equivariant if G acts on E such that for all $g \in G, x \in X, e \in E$, the map $e \mapsto g \cdot e$ is a linear map of vector spaces $E_x \to E_{g \cdot x}$.

A G-equivariant vector bundle produces an ordinary vector bundle $\mathbb{E} \times^G E \to \mathbb{E} \times^G X$.

Definition

Define the **equivariant Chern class** of E as

$$c_k^G(E) \coloneqq c_k(\mathbb{E} \times^G E) \in H_G^{2k}(X).$$

Equivariant Fundamental Class

Similarly, a G-equivariant subvariety $V \subset X$ of codimension d determines a subvariety $\mathbb{E} \times^G V \subset \mathbb{E} \times^G X$ of codimension d.

Definition

Define the **equivariant fundamental class** of V as

$$[V]^G = [\mathbb{E} \times^G V] \in H_G^{2d}(X).$$

Localization

From now on, suppose X is a nonsingular projective T-variety with finitely many fixed points.

Theorem (Atiyah-Bott localization formula)

For any class $\alpha \in H_T^* X$, we have

$$\int_X \alpha = \sum_{p \in X^T} \frac{\alpha|_p}{c^T_{\textit{top}}(T_p X)}.$$

An Important Example

Example (Chern classes of tangent spaces of $\mathbb{P}^{n-1}$)

- Consider $T = (\mathbb{C}^*)^n$ acting on $\mathbb{P}^{n-1}$ by weights $t_1, \ldots, t_n$, i.e.
$$w \cdot [z_1, \ldots z_n] = [t_1(w)z_1, \ldots, t_n(w)z_n],$$
where $t_i \in \mathrm{Hom}(T, C^*)$.
- The fixed points are $p_i = [0, \ldots, 0, 1, 0, \ldots, 0]$ and the weights on $T_{p_i}\mathbb{P}^{n-1}$ are $t_j - t_i$ for $j \neq i$.
- Then
$$c_{n-1}^T(T_{p_i}\mathbb{P}^{n-1}) = \prod_{j \neq i}(t_j - t_i).$$

Steiner's Problem

What is the number of conics that are tangent to five fixed general conics? (Jakob Steiner, 1848)

Conics are defined by homogeneous equations $aX^2 + bY^2 + cZ^2 + dXY + eYZ + fXZ = 0$ modulo scalar multiplication.

Taking the coefficients, the moduli space of conics is $\mathbb{P}(V)$, where $V = \mathrm{Sym}^2 \mathbb{C}^3$.

Then V is 6-dimensional, so $\mathbb{P}(V) \cong \mathbb{P}^5$.

For a fixed conic C, the set of conics tangent to C forms a sextic hypersurface $Z_C \subset \mathbb{P}(V)$. So $[Z_C] = 6H$.

Solution Setup

Steiner's claim: by Bezout's Theorem,

$$\int_{\mathbb{P}(V)} [Z_{C_1}] \cdots [Z_{C_5}] = \int_{\mathbb{P}(V)} (6H)^5 = 7776.$$

Unfortunately, this is **incorrect**.

Indeed, every double line is tangent to any conic, so $\bigcap Z_{C_i}$ contains the Veronese surface $S \subset \mathbb{P}^5$.

(S corresponds to the set of double lines in $\mathbb{P}^2$.)

But $S \cong \mathbb{P}^2$, so $\bigcap Z_{C_i}$ is **not transverse**.

Instead, we need to modify the moduli space by blowing up $\mathbb{P}(V)$ along the Veronese surface S to get the space of complete conics $\mathscr{C} = \mathrm{Bl}_S \, \mathbb{P}(V)$.

Complete Conics

First, given a nonsingular conic $C \subset \mathbb{P}^2$, the (nonsingular) dual curve $C^* \subset (\mathbb{P}^2)^* = \{l \subset \mathbb{P}^2\}$ is

$$C^* := \{l \subset \mathbb{P}^2 : l \text{ is tangent to } C\}.$$

For singular conics, the dual is the closure of the tangents at nonsingular points.

Mapping $[C]$ to $[C^*]$ gives a rational map $\mathbb{P}(V) \dashrightarrow \mathbb{P}(V^*)$ that is regular on $\mathbb{P}(V) \setminus S$.

Complete Conics (cont.)

Its indeterminacy is resolved by blowing up $\mathbb{P}(V)$ at S, and we get

$$\begin{array}{ccc} \mathscr{C} & & \\ \| & & \\ \mathrm{Bl}_S\,\mathbb{P}(V) & = & \overline{\{([C],[C^*]) : C, C^* \text{ smooth}\}} \subset \mathbb{P}(V)\times\mathbb{P}(V^*) \\ \downarrow \pi & \searrow \varphi & \\ \mathbb{P}(V) & \dashrightarrow & \mathbb{P}(V^*) \end{array}$$

Characterization of Complete Conics

One can show that $\mathscr{C}$ consists of four types of figures in $\mathbb{P}^2$:

1. nonsingular conics
2. unions of two lines
3. lines with two marked points
4. lines with one marked point

Solving Steiner's Problem

Let $\widetilde{Z_C}$ be the proper transform of Z_C.

Given five general conics $C_1, \ldots, C_5$, one can prove that $\widetilde{Z_{C_i}}$ intersect transversally.

For $p \in \mathbb{P}^2$, define hyperplane $\Sigma_p \subset \mathbb{P}(V)$ of conics containing p.

For $l \in (\mathbb{P}^2)^*$, define hyperplane $\Theta_l \subset \mathbb{P}(V^*)$ of conics tangent to l.

Let σ_p, τ_l be the proper transforms. Non-equivariantly, their classes are independent of p and l.

One can show that $\widetilde{Z} = 2\sigma + 2\tau$ in $H^*\mathscr{C}$.

The Correct Integral

We therefore want to calculate

$$\int_{\mathscr{C}} [\widetilde{Z}]^5 = \int_{\mathscr{C}} (2\sigma + 2\tau)^5$$
$$= 32\Big(\int \sigma^5 + 5 \int \sigma^4\tau + 10 \int \sigma^3\tau^2 + 10 \int \sigma^2\tau^3 + 5 \int \sigma\tau^4 + \int \tau^5 \Big).$$

Note that

- $\int \sigma^i \tau^{5-i} = \int \sigma^{5-i}\tau^i$ by symmetry;
- $\int \sigma^5 = 1$.

So we only need to calculate $\int \sigma^4\tau$ and $\int \sigma^3\tau^2$.

Constructing the Torus Action

Let $T = (\mathbb{C}^*)^3$ act on $\mathbb{P}^2$ with weights t_1, t_2, t_3.

The induced action on $V = \operatorname{Sym}^2 \mathbb{C}^3$ has weights

$$2t_1, 2t_2, 2t_3, t_1 + t_2, t_2 + t_3, t_1 + t_3$$

corresponding to the basis

$$X^2, Y^2, Z^2, XY, YZ, XZ.$$

The tangent spaces to $\mathbb{P}(V)$ at X^2 and XY have weights

$$T_{X^2}\mathbb{P}(V)\colon \{2(t_2 - t_1), 2(t_3 - t_1), t_2 - t_1, t_2 + t_3 - 2t_1, t_3 - t_1\}$$
$$T_{XY}\mathbb{P}(V)\colon \{t_1 - t_2, t_2 - t_1, 2t_3 - t_1 - t_2, t_3 - t_1, t_3 - t_2\}.$$

Fixed Points of the Torus Action

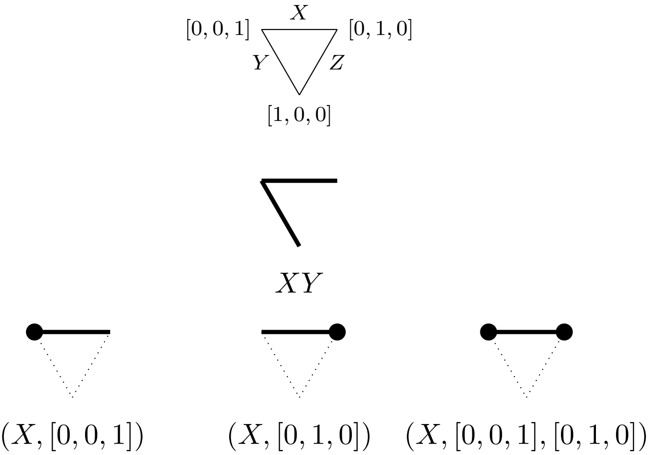
Recall the localization formula:

$$\int_X \alpha = \sum_{p \in X^T} \frac{\alpha|_p}{c^T_{\mathsf{top}}(T_pX)}.$$

To use localization, we need the **fixed points** of $\mathscr{C}$ and their **tangent weights.** Note that $\pi\colon \mathscr{C} \to \mathbb{P}(V)$ maps fixed points to fixed points.

1. Over XY, π is an isomorphism, so the corresponding fixed point in $\mathscr{C}$ is a pair of lines.
2. Over X^2, we need to add data of one or two points (which also need to be T-invariant).

Helpful Diagrams



Tangent Weights of a Blowup

Theorem

Let X be a smooth T-variety and $S \subset X$ a smooth T-invariant subvariety. Let $\widetilde{X} = \mathrm{Bl}_S X$ be the blowup, $N = N_{S/X}$ the normal bundle, and $E = \mathbb{P}(N) \to S$ the exceptional divisor. Take $x \in E$, which corresponds to a line $L \subset N_s$, i.e. $L = \mathscr{O}(-1)|_x$, where $E \to S$ maps x to s. Then

$$\begin{aligned} T_x\widetilde{X} &= T_xE \oplus L \\ &= \mathrm{Hom}(L, N_s/L) \oplus T_sS \oplus L. \end{aligned}$$

Using this, let us calculate the weights at T$\mathscr{C}$.

Calculating the Tangent Weights (1)

Recall that  projects to $X^2 \in S \subset \mathbb{P}(V)$ and that $T_{X^2}\mathbb{P}(V)$ has weights

$$T_{X^2}\mathbb{P}(V)\colon \{2(t_2 - t_1), 2(t_3 - t_1), t_2 - t_1, t_2 + t_3 - 2t_1, t_3 - t_1\}.$$

From the action of T on $S \cong \mathbb{P}^2$, the tangent space $T_{X^2}S$ has weights

$$T_{X^2}S\colon \{t_2 - t_1, t_3 - t_1\}.$$

So the normal weights to S at X^2 are

$$N_{X^2}S\colon \{2(t_2 - t_1), 2(t_3 - t_1), t_2 + t_3 - 2t_1\}.$$

Calculating the Tangent Weights (2)

Each weight corresponds to a T-invariant line $L \subset N$, hence a T-fixed point in E.

$2(t_2 - t_1) \leftrightarrow$ 

$2(t_1 - t_2) \leftrightarrow$ 

$t_2 + t_3 - 2t_1 \leftrightarrow$ 

Take $L \leftrightarrow 2(t_2 - t_1) \leftrightarrow$ .

Calculating the Tangent Weights (3)

We have

T$E\colon \{t_2 - t_1, t_3 - t_1\}$

$$\cup \{(t_2 + t_3 - 2t_1) - 2(t_2 - t_1), 2(t_3 - t_1) - 2(t_2 - t_1)\}$$

and so

T$\mathscr{C}\colon \{t_2 - t_1, t_3 - t_1, t_3 - t_2, 2(t_3 - t_2)\} \cup \{2(t_2 - t_1)\}.$

Similar calculations for  and .

The Hyperplane Classes

Recall the localization formula:

$$\int_X \alpha = \sum_{p \in X^T} \frac{\alpha|_p}{c^T_{\text{top}}(T_p X)}.$$

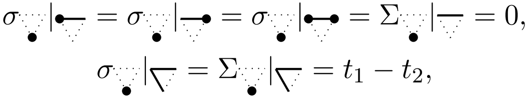
We need the restrictions of σ_p and τ_l at each fixed point.

One can show that

$$\sigma_p = \pi^* \Sigma_p,$$
$$\tau_l = \varphi^* \Sigma_l^*,$$
$$[\Sigma_p]^T = c_1^T(\mathscr{O}(1) \otimes \mathbb{C}_{2t_1})$$

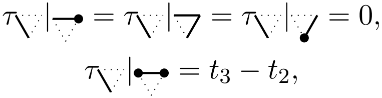
Restrictions of the Classes at the Fixed Points

So for $p =$ ,



etc.

and for $l =$ ,



etc.

Calculating the Integrals

To calculate $\int_{\mathscr{C}} \sigma^4 \tau$, consider the class

$$\alpha = (\sigma_{\nabla})^2 \sigma_{\nabla} \sigma_{\nabla} \tau_{\nabla}.$$

Then α_{∇} and $\alpha_{\bigvee}$ are the only nonzero restrictions. So

$$\begin{aligned}
\int_{\mathscr{C}} \sigma^4 \tau &= \int_{\mathscr{C}} \alpha \\
&= \frac{\alpha|_{\nabla}}{c^T_{\text{top}}(T_{\nabla}\mathscr{C})} + \frac{\alpha|_{\bigvee}}{c^T_{\text{top}}(T_{\bigvee}\mathscr{C})} \\
&= \frac{\alpha|_{\nabla}}{\prod\{\text{weights of } T_{\nabla}\mathscr{C})\}} + \frac{\alpha|_{\bigvee}}{\prod\{\text{weights of } T_{\bigvee}\mathscr{C})\}} \\
&= \cdots \\
&= 2.
\end{aligned}$$

Final Solution

Do a similar calculation for $\sigma^3\tau^2$ to get $\int_{\mathscr{C}} \sigma^3\tau^2 = 4$.

We then have

$$\begin{aligned}
\int_{\mathscr{C}} [\widetilde{Z}]^5 &= \int_{\mathscr{C}} (2\sigma + 2\tau)^5 \\
&= 32\bigg(\int \sigma^5 + 5\int \sigma^4\tau + 10\int \sigma^3\tau^2 \\
&\qquad + 10\int \sigma^2\tau^3 + 5\int \sigma\tau^4 + \int \tau^5 \bigg) \\
&= 32(1 + 5\cdot 2 + 10\cdot 4 + 10\cdot 4 + 5\cdot 2 + 1) \\
&= 3264.
\end{aligned}$$

This is the correct solution to Steiner's problem.